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Invariant multidimensional matrices

Jean Vallès

"On se persuade mieux, pour l’ordinaire, par les raisons qu’on a soi-même trouvées que par celles qui sont venues dans l’esprit des autres" (Pascal).

Abstract: In [AO] the authors study Steiner bundles via their unstable hyperplanes and proved that (see [AO], Tmm 5.9):  

A rank $n$ Steiner bundle on $\mathbb{P}^n$ which is $SL(2,\mathbb{C})$ invariant is a Schwarzenberger bundle.

In this note we give a very short proof of this result based on Clebsch-Gordon problem for $SL(2,\mathbb{C})$-modules.

1 Introduction

Let $A$, $B$, and $C$ three vector spaces over $\mathbb{C}$ and $\phi : A \otimes B \to C^*$ a linear surjective map. We consider the sheaf $S_\phi$ on $\mathbb{P}(A) \times \mathbb{P}(B)$ defined by,

$$0 \longrightarrow S_\phi \longrightarrow C \otimes O_{\mathbb{P}(A) \times \mathbb{P}(B)} \xrightarrow{\phi} O_{\mathbb{P}(A) \times \mathbb{P}(B)}(1,1)$$

with fibers $S_\phi(a \otimes b) = \{c \in C \mid \phi(a \otimes b)(c) = 0\}$.

Remark 1. If $\dim \mathbb{C}C < \dim \mathbb{C}A + \dim \mathbb{C}B - 1$ then $\ker(\phi)$ meets the set of decomposable tensors, so there exist $a \in A$, $a \neq 0$ and $b \in B$, $b 
eq 0$ such that $\phi(a \otimes b) = 0$.

Remark 2. When $S_\phi$ is a vector bundle it gives two “associated” Steiner bundles $S_A$ on $\mathbb{P}(A)$ and $S_B$ on $\mathbb{P}(B)$ after projections (see [DK], prop. 3.20).

We denote by $(\mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C))^\vee$ the variety of hyperplanes tangent to the Segre $\mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C)$ and by $(\mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C))^\vee$ the set of hyperplanes in $\mathbb{P}(A \otimes B \otimes C)$ containing $\{a\} \times \{b\} \times \mathbb{P}(C)$.

The next proposition is a reformulation of many results from, [GKZ] (see for instance Thm 3.1’ page 458, prop 1.1 page 445), [AO] (see thm page 1) and [DO] (see cor.3.3).

Proposition 1.1. Let $A$, $B$, and $C$ three vector spaces over $\mathbb{C}$ with $\dim \mathbb{C}A = n + 1$, $\dim \mathbb{C}B = m + 1$ and $\dim \mathbb{C}C \geq n + m + 1$ and $\phi : A \otimes B \to C^*$ a surjective linear map. Then the following propositions are equivalent:

1) $S_\phi$ is a vector bundle over $\mathbb{P}(A) \times \mathbb{P}(B)$.
2) $\phi(a \otimes b) \neq 0$ for all $a \in A$, $a \neq 0$ and $b \in B$, $b \neq 0$.
3) $\phi \notin (\mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C))^\vee$ for all $a \in A$, $a \neq 0$ and $b \in B$, $b \neq 0$.
4) $\phi \notin (\mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C))^\vee$

Remark 3. When $\dim \mathbb{C}C = \dim \mathbb{C}A + \dim \mathbb{C}B - 1$ the variety $(\mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C))^\vee$ is an hypersurface in $\mathbb{P}((A \otimes B \otimes C)^\vee)$. This hypersurface is defined by the vanishing of the
hyperdeterminant, say $\text{Det}(\Phi)$ where $\Phi$ is the generic tridimensional matrix (see [GKZ], chapter 1 and 14).

**Proof.** It is clear that 1) 2) and 3) are equivalent. It remains to show that 3) and 4) are equivalent too.

Since $\partial(abc) = (\partial(a)bc + a(\partial(b))c + ab(\partial(c)))$ an hyperplane $H$ is tangent to the Segre in a point $(a, b, c)$ if and only if it contains $\mathbb{P}(A) \times \{b\} \times \{c\}$ and $\{a\} \times \mathbb{P}(B) \times \{c\}$ and $\{a\} \times \{b\} \times \mathbb{P}(C)$. We prove here that the third condition implies the two others. Let $H$ an hyperplane containing $\{a\} \times \{b\} \times \mathbb{P}(C)$, we show that there exists $c \in C$ such that $H$ contains $\mathbb{P}(A) \times \{b\} \times \{c\}$ and $\{a\} \times \mathbb{P}(B) \times \{c\}$. Let $\phi$ the trilinear application corresponding to $H$. Since $\phi(a \otimes b)(C) = 0$ we have a $\dim \mathbb{C}C$-dimensional family of bilinear forms vanishing on $(a, b)$. Now finding a bilinear form of the above family (i.e. finding $c \in C$) which verify $\phi(A \otimes b)(c) = 0$ and $\phi(a \otimes B)(c) = 0$ imposes at most $n + m$ conditions. Since $\dim \mathbb{C}C \geq n + m + 1$, this point $c$ exists. $\square$

2 Invariant tridimensional matrix under $SL(2, \mathbb{C})$-action

In the second part of this note we will consider the boundary case

$$\dim \mathbb{C}C = \dim \mathbb{C}A + \dim \mathbb{C}B - 1$$

Then, instead of writing $\phi$ induces a vector bundle or $\phi \notin (\mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C))^\vee$ we will write equivalently $\text{Det}(\phi) \neq 0$.

We denote by $S_i$ the irreducible $SL(2, \mathbb{C})$-representations of degree $i$ and by $(x^{i-k} y^k)_{k=0, \ldots, i}$ a basis of $S_i$.

**Theorem 2.1.** Let $A$, $B$ and $C$ be three non trivial $SL(2, \mathbb{C})$-modules with dimension $n+1$, $m+1$ and $n+m+1$ and $\phi \in \mathbb{P}(A \otimes B \otimes C)$ an invariant hyperplane under $SL(2, \mathbb{C})$. Then,

$$\text{Det}(\phi) \neq 0 \Leftrightarrow \phi$$

is the multiplication $S_n \otimes S_m \rightarrow S_{n+m}$.

**Proof.** When $\phi \in \mathbb{P}(S_n \otimes S_m \otimes S_{n+m})$ is just the multiplication $S_n \otimes S_m \rightarrow S_{n+m}$ it is well known that it corresponds to Schwarzenberger bundles (see [DK], prop 6.3).

Conversely, let $A = \oplus_{i \in I} S_i \otimes U_i$, $B = \oplus_{j \in J} S_j \otimes V_j$ where $U_i$, $V_j$ are trivial $SL(2, \mathbb{C})$-representations of dimension $n_i$ and $m_j$. Let $x^i \in S_i$, $x^j \in S_j$ be two highest weight vectors and $u \in U_i$, $v \in V_j$. Since $\text{Det}(\phi) \neq 0$, $\phi((x^i \otimes u) \otimes (x^j \otimes v)) \neq 0$ and by $SL(2, \mathbb{C})$-invariance $\phi((x^i \otimes u) \otimes (x^j \otimes v)) = x^{i+j} \phi(u \otimes v) \in S_{i+j} \otimes W_{i+j}$. By hypothesis $\phi(u \otimes v) \neq 0$ for all $u \in U_i$ and $v \in V_j$ so, by the Remark 1, it implies that $\dim W_{i+j} \geq n_i + m_j - 1$, and $S_{i+j}^{n_i+m_j-1} \subset C^*$.

Assume now that $B$ contains at least two distinct irreducible representations. Let $i_0$ and $j_0$ the greatest integers in $I$ and $J$. We consider the submodule $B_1$ such that $B_1 \oplus S_{j_0}^{m_{j_0}} = B$.

Then the restricted map $A \otimes B_1 \rightarrow C^*$ is not surjective because the image is concentrated in the submodule $C^*_1$ of $C^*$ defined by $C_1^* \oplus S_{i_0+j_0}^{m_{i_0}+m_{j_0}} = C^*$. Now since

$$\dim \mathbb{C}(C_1^*) = \dim \mathbb{C}(A) + \dim \mathbb{C}(B_1) - 1$$

there exist $a \in A$, $b \in B_1 \subset B$ such that $\phi(a \otimes b) = 0$. A contradiction with the hypothesis $\text{Det}(\phi) \neq 0$.


So \( A = S_i^n \), \( B = S_j^m \) and \( S_{i+j}^{n_i+m_j-1} \subset C^* \). Since \( \dim C = \dim A + \dim B - 1 \), we have \((i+1)n_i + (j+1)m_j - 1 = \dim C \geq (i+j+1)(n_i + m_j - 1) \) which is possible if and only if \( n_i = m_j = 1 \) and \( C = S_{i+j} \). \( \Box \)

**Corollary 2.2.** A rank \( n \) Steiner bundle on \( \mathbb{P}^n \) which is \( SL(2, \mathbb{C}) \) invariant is a Schwarzenberger bundle.

**Proof.** Let \( S \) a rank \( n \) Steiner bundle on \( \mathbb{P}^n \), i.e \( S \) appears in an exact sequence

\[
0 \to S \to C \otimes O_{\mathbb{P}(A)} \to B^* \otimes O_{\mathbb{P}(A)}(1) \to 0
\]

where \( \mathbb{P}(A) = \mathbb{P}^n \), \( \mathbb{P}(B) = \mathbb{P}^m \) and \( \mathbb{P}(C) = \mathbb{P}^{n+m} \). If \( SL(2, \mathbb{C}) \) acts on \( S \) the vector spaces \( A \), \( B \) and \( C \) are \( SL(2, \mathbb{C}) \)-modules since \( A \) is the basis, \( B^* = H^1S(-1) \) and \( C^* = H^0(S^*) \). If \( S \) is \( SL(2, \mathbb{C}) \)-invariant the linear surjective map

\[
A \otimes (H^1S(-1))^* \to H^0(S^*)
\]

is \( SL(2, \mathbb{C}) \)-invariant too. \( \Box \)

**Remark.** The proofs of the theorem and the proposition, given in this paper, are still valid for more than three vector spaces when the format is the boundary format.

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**References**


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